Three-level atoms

The rotating-wave approximation, dark states, EIT, slow light and Raman transitions

Michał Parniak · Centre of New Technologies, University of Warsaw
m.parniak@cent.uw.edu.pl · qodl.eu/teaching

About these notes

One idea runs through this lecture: *two paths to the same excited state can interfere and cancel*. Part I shows this in the simplest possible way, a dark state. Part II shows what it does to light: a medium that should absorb becomes transparent, light slows down, and far from resonance the atom behaves like a two-level system made of its two ground states.

What you need: the Schrödinger equation for amplitudes and complex numbers. Lasers are treated as classical fields; only the atom is quantum. No density matrices: we use amplitudes throughout and add losses by hand. $\hbar = 1$, so energies and frequencies are the same thing.

Contents

Part I — The dark state 2

1 A two-level atom driven by a laser, and the rotating-wave approximation 2

2 The Λ system and its Hamiltonian 3

3 The dark state 5

4 Coherent population trapping 5

Part II — What it does to light 6

5 A weak probe: the steady state 6

6 Electromagnetically induced transparency 6

7 Slow light 7

8 Far from resonance: Raman transitions 8

Summary 8

Part I — The dark state

1 A two-level atom driven by a laser, and the rotating-wave approximation

Hamiltonian. The atom has levels $|g\rangle$ (energy 0) and $|e\rangle$ (energy ω_{eg}). The laser field at the atom is $E(t) = E_0 \cos\omega t$, and it couples through the dipole operator, $-\hat{d}\,E(t)$, with $\hat{d} = d\,(|e\rangle\langle g| + |g\rangle\langle e|)$ and d real. Defining the **Rabi frequency**

$$\Omega \equiv \frac{dE_0}{\hbar},$$

the full Hamiltonian is

$$\hat{H} = \omega_{eg}|e\rangle\langle e| - \Omega\cos(\omega t)\,(|e\rangle\langle g| + |g\rangle\langle e|).$$

No approximation has been made yet (other than keeping two levels).

Step 1: remove the free evolution. Without the laser, $|e\rangle$ simply picks up a phase $\mathrm{e}^{-\mathrm{i}\omega_{eg}t}$. We take this phase out by hand and write

$$|\psi\rangle = b_g|g\rangle + b_e\,\mathrm{e}^{-\mathrm{i}\omega_{eg}t}|e\rangle.$$

Insert into $\mathrm{i}\partial_t|\psi\rangle = \hat{H}|\psi\rangle$. On the left, the $|e\rangle$ component is $(\mathrm{i}\dot{b}_e + \omega_{eg}b_e)\mathrm{e}^{-\mathrm{i}\omega_{eg}t}$; on the right it is $\omega_{eg}b_e\mathrm{e}^{-\mathrm{i}\omega_{eg}t} - \Omega\cos(\omega t)\,b_g$. The terms $\omega_{eg}b_e$ cancel, and after multiplying by $\mathrm{e}^{\mathrm{i}\omega_{eg}t}$:

$$\mathrm{i}\dot{b}_e = -\Omega\cos(\omega t)\,\mathrm{e}^{\mathrm{i}\omega_{eg}t}\,b_g, \qquad \mathrm{i}\dot{b}_g = -\Omega\cos(\omega t)\,\mathrm{e}^{-\mathrm{i}\omega_{eg}t}\,b_e.$$

Now the amplitudes change only because of the laser.

Step 2: two kinds of terms. Write $\cos\omega t = \frac{1}{2}(\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{e}^{\mathrm{i}\omega t})$ and define the detuning $\Delta \equiv \omega - \omega_{eg}$:

Exact equations, before the RWA

$$\mathrm{i}\dot{b}_e = -\frac{\Omega}{2}\Big(\underbrace{\mathrm{e}^{-\mathrm{i}\Delta t}}_{\text{slow}} + \underbrace{\mathrm{e}^{\mathrm{i}(\omega+\omega_{eg})t}}_{\text{fast}}\Big)\,b_g, \qquad \mathrm{i}\dot{b}_g = -\frac{\Omega}{2}\Big(\mathrm{e}^{\mathrm{i}\Delta t} + \mathrm{e}^{-\mathrm{i}(\omega+\omega_{eg})t}\Big)\,b_e.$$

For a laser close to resonance, Δ is at most a few MHz, while $\omega + \omega_{eg} \approx 2\omega$ is an optical frequency, $\sim 10^{15}\,\mathrm{s}^{-1}$. The first term is called *co-rotating*, the second *counter-rotating*.

Step 3: how big is each term? Start in $|g\rangle$ and look at short times, so that $b_g \approx 1$ on the right-hand side. Integrate:

$$b_e(t) = \frac{\mathrm{i}\Omega}{2}\int_0^t \mathrm{d}t'\,\Big(\mathrm{e}^{-\mathrm{i}\Delta t'} + \mathrm{e}^{\mathrm{i}(\omega+\omega_{eg})t'}\Big) = \frac{\mathrm{i}\Omega}{2}\left[\frac{1-\mathrm{e}^{-\mathrm{i}\Delta t}}{\mathrm{i}\Delta} + \frac{\mathrm{e}^{\mathrm{i}(\omega+\omega_{eg})t}-1}{\mathrm{i}(\omega+\omega_{eg})}\right].$$

- The co-rotating term: for $\Delta = 0$ it equals $\frac{\mathrm{i}\Omega}{2}\,t$. It *grows* steadily: this is what transfers the atom to $|e\rangle$.
- The counter-rotating term: its size never exceeds $\frac{\Omega}{\omega+\omega_{eg}}$. It wiggles back and forth once per optical cycle, about every femtosecond, and averages to zero.

Numbers for rubidium: $\omega/2\pi \approx 384\,\mathrm{THz}$ and a typical $\Omega/2\pi = 10\,\mathrm{MHz}$ give $\Omega/(\omega+\omega_{eg}) \approx 10^{-8}$.

The rotating-wave approximation (RWA)

Drop the terms oscillating at $\omega + \omega_{eg}$. This is justified when

$$\Omega,\ |\Delta| \ \ll\ \omega + \omega_{eg} \approx 2\omega.$$

The neglected terms cause tiny corrections, for example a shift of the resonance of order Ω^2/ω (the Bloch–Siegert shift), below 1 Hz in the example above. Going back to the lab frame, the RWA Hamiltonian is

$$\hat{H}_{\mathrm{RWA}} = \omega_{eg}|e\rangle\langle e| - \frac{\Omega}{2}\Big(\mathrm{e}^{-\mathrm{i}\omega t}|e\rangle\langle g| + \mathrm{e}^{\mathrm{i}\omega t}|g\rangle\langle e|\Big).$$

Two ways to remember which term stays.

- *Energy.* $\mathrm{e}^{-\mathrm{i}\omega t}|e\rangle\langle g|$ takes the atom up *and* takes energy ω from the field (absorption): energy is conserved up to Δ. The dropped term $\mathrm{e}^{+\mathrm{i}\omega t}|e\rangle\langle g|$ takes the atom up while *giving* energy ω to the field. It misses energy conservation by $\omega + \omega_{eg}$, so it can act only for a time $\sim 1/(\omega + \omega_{eg})$. (In the Jaynes–Cummings model the same two terms are $\hat{a}\,|e\rangle\langle g|$, kept, and $\hat{a}^\dagger\,|e\rangle\langle g|$, dropped.)
- *Rotation.* A linearly oscillating field is the sum of two fields rotating in opposite senses at frequency ω. The atomic dipole rotates in one sense at ω_{eg}. The co-rotating component keeps a fixed phase relative to the dipole and acts cumulatively; the counter-rotating component sees the dipole at frequency $\omega + \omega_{eg}$ and its effect averages out. Hence the name.

Step 4: the rotating frame. After the RWA the equations still contain $\mathrm{e}^{\mp\mathrm{i}\Delta t}$. We remove it with one more phase, $b_e = c_e\,\mathrm{e}^{-\mathrm{i}\Delta t}$. Then $\mathrm{i}\dot{b}_e = (\mathrm{i}\dot{c}_e + \Delta c_e)\,\mathrm{e}^{-\mathrm{i}\Delta t}$, and

$$\mathrm{i}\dot{c}_g = -\frac{\Omega}{2}\,c_e, \qquad \mathrm{i}\dot{c}_e = -\Delta\,c_e - \frac{\Omega}{2}\,c_g.$$

Together, the two phases mean that we wrote

$$|\psi\rangle = c_g|g\rangle + c_e\,\mathrm{e}^{-\mathrm{i}\omega t}|e\rangle\ :$$

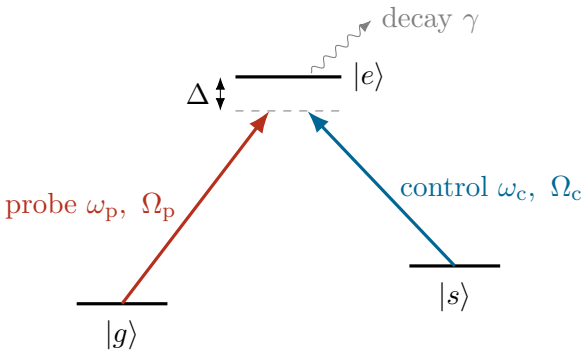
the excited-state amplitude rotates with the *laser*, not with the atom. This is the **rotating frame**. Unlike the RWA, this step is exact: it is just a change of variables, and all time dependence is gone. The equations are those of the time-independent Hamiltonian

$$\hat{H} = -\begin{pmatrix} 0 & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \Delta \end{pmatrix} \qquad \text{in the basis } (|g\rangle, |e\rangle).$$

On resonance, $\Delta = 0$, with $c_g(0) = 1$: $c_g = \cos\frac{\Omega t}{2}$, $c_e = \mathrm{i}\sin\frac{\Omega t}{2}$. These are **Rabi oscillations**, $P_e = \sin^2(\Omega t/2)$, and Ω measures how strongly the laser couples the two levels.

2 The Λ system and its Hamiltonian

Now add a third level. In alkali atoms the ground state is split into two hyperfine levels, $|g\rangle$ and $|s\rangle$, a few GHz apart. Both live practically for ever. They share an excited state $|e\rangle$. Two lasers: the **probe** (frequency ω_p) is meant to drive $|g\rangle \leftrightarrow |e\rangle$, and the **control** (frequency ω_c) is meant to drive $|s\rangle \leftrightarrow |e\rangle$.



The shape of the diagram gives the name: a Λ system. Energies are 0, ω_{sg}, ω_{eg} for $|g\rangle, |s\rangle, |e\rangle$.

The RWA, now with two lasers. The full field $E_\mathrm{p}\cos\omega_\mathrm{p}t + E_\mathrm{c}\cos\omega_\mathrm{c}t$ couples to both transitions. After removing the free evolution (Step 1 above), every coupling term oscillates at the laser frequency minus or plus the transition frequency. With $\omega_{es} = \omega_{eg} - \omega_{sg}$ and the detunings Δ, δ defined in the next paragraph:

transition	laser	co-rotating frequency		counter-rotating	
$g \leftrightarrow e$	probe	$\omega_\mathrm{p} - \omega_{eg} = \Delta$	keep	$\omega_\mathrm{p} + \omega_{eg}$	drop
$s \leftrightarrow e$	control	$\omega_\mathrm{c} - \omega_{es} = \Delta - \delta$	keep	$\omega_\mathrm{c} + \omega_{es}$	drop
$g \leftrightarrow e$	control	$\omega_\mathrm{c} - \omega_{eg} \approx -\omega_{sg}$	drop	$\omega_\mathrm{c} + \omega_{eg}$	drop
$s \leftrightarrow e$	probe	$\omega_\mathrm{p} - \omega_{es} \approx +\omega_{sg}$	drop	$\omega_\mathrm{p} + \omega_{es}$	drop

The counter-rotating terms are dropped exactly as in Section 1. The last two rows are the "wrong" laser on each transition. They oscillate at about the ground-state splitting ω_{sg} ($2\pi \times 6.8\,\mathrm{GHz}$ in ^{87}Rb), which is again much larger than Ω, Δ and γ (MHz), so they average out by the same argument. This second approximation is weaker ($\Omega/\omega_{sg} \sim 10^{-3}$ instead of 10^{-8}). When the two ground states are close in energy, for example Zeeman sublevels, one uses polarization selection rules instead to make each laser drive only its own transition. Keeping the near-resonant terms, with Ω_p and Ω_c the two Rabi frequencies:

$$\hat{H} = \omega_{sg}|s\rangle\langle s| + \omega_{eg}|e\rangle\langle e| - \frac{\Omega_\mathrm{p}}{2}\left(\mathrm{e}^{-\mathrm{i}\omega_\mathrm{p}t}|e\rangle\langle g| + \mathrm{h.c.}\right) - \frac{\Omega_\mathrm{c}}{2}\left(\mathrm{e}^{-\mathrm{i}\omega_\mathrm{c}t}|e\rangle\langle s| + \mathrm{h.c.}\right).$$

The rotating frame. We need a phase factor for each level. The rule: going up with the probe costs $\mathrm{e}^{-\mathrm{i}\omega_\mathrm{p}t}$, going up with the control costs $\mathrm{e}^{-\mathrm{i}\omega_\mathrm{c}t}$. To reach $|e\rangle$ from $|g\rangle$ we use the probe; to reach $|s\rangle$ from $|g\rangle$ we go up with the probe and down with the control. So

$$|\psi\rangle = c_g|g\rangle + c_e\,\mathrm{e}^{-\mathrm{i}\omega_\mathrm{p}t}|e\rangle + c_s\,\mathrm{e}^{-\mathrm{i}(\omega_\mathrm{p}-\omega_\mathrm{c})t}|s\rangle.$$

Insert into $\mathrm{i}\partial_t|\psi\rangle = \hat{H}|\psi\rangle$ and compare coefficients:

$$\begin{aligned}
\mathrm{i}\dot{c}_g &= -\tfrac{\Omega_\mathrm{p}}{2}\,c_e,\\
\mathrm{i}\dot{c}_e &= -\Delta\,c_e - \tfrac{\Omega_\mathrm{p}}{2}\,c_g - \tfrac{\Omega_\mathrm{c}}{2}\,c_s,\\
\mathrm{i}\dot{c}_s &= -\delta\,c_s - \tfrac{\Omega_\mathrm{c}}{2}\,c_e,
\end{aligned}
\qquad
\begin{aligned}
\Delta &\equiv \omega_\mathrm{p} - \omega_{eg} \quad \text{(one-photon detuning)},\\
\delta &\equiv \omega_\mathrm{p} - \omega_\mathrm{c} - \omega_{sg} \quad \text{(two-photon detuning)}.
\end{aligned}$$

As an example, check the c_s equation. The left side gives $\mathrm{i}\dot{c}_s + (\omega_\mathrm{p} - \omega_\mathrm{c})c_s$, times the phase factor. On the right we have $\omega_{sg}c_s$, and the control term $-\frac{\Omega_\mathrm{c}}{2}\mathrm{e}^{\mathrm{i}\omega_\mathrm{c}t}|s\rangle\langle e|$ acting on $c_e\mathrm{e}^{-\mathrm{i}\omega_\mathrm{p}t}|e\rangle$ gives $-\frac{\Omega_\mathrm{c}}{2}c_e\mathrm{e}^{-\mathrm{i}(\omega_\mathrm{p}-\omega_\mathrm{c})t}$: the same phase factor. It cancels, and $\omega_{sg} - (\omega_\mathrm{p} - \omega_\mathrm{c}) = -\delta$.

The Λ Hamiltonian in the rotating frame

The same equations come from a time-independent matrix. In the basis $(|g\rangle, |e\rangle, |s\rangle)$:

$$\hat{H} = -\begin{pmatrix} 0 & \frac{\Omega_\mathrm{p}}{2} & 0 \\ \frac{\Omega_\mathrm{p}}{2} & \Delta & \frac{\Omega_\mathrm{c}}{2} \\ 0 & \frac{\Omega_\mathrm{c}}{2} & \delta \end{pmatrix}.$$

The two detunings mean different things. Δ says how far the probe is from the excited state. δ says whether the *difference* of the two laser frequencies matches the splitting of the ground states. Most of the physics below happens at $\delta = 0$.

3 The dark state

In the matrix, $|e\rangle$ is reached from $|g\rangle$ with amplitude $\frac{\Omega_\mathrm{p}}{2}$ and from $|s\rangle$ with amplitude $\frac{\Omega_\mathrm{c}}{2}$. If the atom is in a superposition of $|g\rangle$ and $|s\rangle$, the two contributions add, and with the right signs they cancel.

Take $\delta = 0$ and try $|D\rangle = a|g\rangle + b|s\rangle$. Then

$$\langle g|\hat{H}|D\rangle = 0, \qquad \langle s|\hat{H}|D\rangle = 0, \qquad \langle e|\hat{H}|D\rangle = -\tfrac{1}{2}(\Omega_\mathrm{p}\, a + \Omega_\mathrm{c}\, b).$$

This vanishes if $\Omega_\mathrm{p} a + \Omega_\mathrm{c} b = 0$. Normalizing:

Dark and bright states ($\delta = 0$)

$$|D\rangle = \frac{\Omega_\mathrm{c}|g\rangle - \Omega_\mathrm{p}|s\rangle}{\Omega}, \qquad |B\rangle = \frac{\Omega_\mathrm{p}|g\rangle + \Omega_\mathrm{c}|s\rangle}{\Omega}, \qquad \Omega \equiv \sqrt{\Omega_\mathrm{p}^2 + \Omega_\mathrm{c}^2}.$$

$\hat{H}|D\rangle = 0$ for any Δ. $\langle e|\hat{H}|B\rangle = -\frac{\Omega_\mathrm{p}^2+\Omega_\mathrm{c}^2}{2\Omega} = -\frac{\Omega}{2}$, and $\langle D|\hat{H}|B\rangle = 0$. In the basis $(|D\rangle, |B\rangle, |e\rangle)$:

$$\hat{H} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\Omega}{2} \\ 0 & \frac{\Omega}{2} & \Delta \end{pmatrix}.$$

An atom in $|D\rangle$ stays there for ever. It never reaches $|e\rangle$, so it never emits: it is *dark*. The relative sign matters: with a plus sign the two paths would add, which is the bright state.

What interferes here is not the two fields: they have different frequencies and drive different transitions. What interferes are the two *quantum amplitudes* for reaching $|e\rangle$.

The three-level problem has split into two simple pieces: a dark state that does nothing, and an ordinary two-level atom $|B\rangle \leftrightarrow |e\rangle$ with Rabi frequency Ω. Everything from Section 1 applies to the bright part.

A useful limit: the control is much stronger than the probe, $\Omega_\mathrm{c} \gg \Omega_\mathrm{p}$. Then $|D\rangle \approx |g\rangle$. The atom sits in its ground state, the probe is on, and yet *nothing happens*: the control field has made $|g\rangle$ dark for the probe. This is the essence of the transparency in Part II.

4 Coherent population trapping

How does an atom get into $|D\rangle$? Through spontaneous emission. Start anywhere. The bright part is driven up to $|e\rangle$, which decays at rate γ to $|g\rangle$ or $|s\rangle$, landing in some random superposition. Each time, part of it lands in $|D\rangle$, and that part stays. After a few cycles all population is in $|D\rangle$, and the fluorescence stops.

This is **coherent population trapping** (CPT). It was discovered as a dark line in the fluorescence of sodium atoms [1]: the glowing gas went dark exactly where the frequency difference

of the two lasers matched the ground-state splitting, $\delta = 0$. Because it is an interference effect between long-lived ground states, the dark line is very narrow, often a few kHz, even though the excited state is $\sim 6\,\mathrm{MHz}$ wide. This is used in compact atomic clocks and magnetometers.

Part II — What it does to light

5 A weak probe: the steady state

We send a weak probe through a gas of such atoms, with the control on. Does the gas absorb the probe? The atoms start in $|g\rangle$. The probe is weak, so they stay there: $c_g \approx 1$.

We add losses by hand, as decay of amplitudes. $|e\rangle$ decays at rate γ, so its amplitude decays at $\gamma/2$. The g–s coherence decays slowly, at $\gamma_s/2$ (collisions, magnetic-field noise):

$$\begin{aligned} \mathrm{i}\dot{c}_e &= -(\Delta + \mathrm{i}\tfrac{\gamma}{2})c_e - \tfrac{\Omega_\mathrm{p}}{2} - \tfrac{\Omega_\mathrm{c}}{2}\,c_s, \\ \mathrm{i}\dot{c}_s &= -(\delta + \mathrm{i}\tfrac{\gamma_s}{2})c_s - \tfrac{\Omega_\mathrm{c}}{2}\,c_e. \end{aligned}$$

(Check the sign: with $\Delta = \delta = 0$ and no coupling, $\mathrm{i}\dot{c}_e = -\mathrm{i}\frac{\gamma}{2}c_e$ gives $c_e \propto \mathrm{e}^{-\gamma t/2}$.) The probe drives, the losses damp, and after a time $\sim 1/\gamma$ the amplitudes stop changing. Setting the time derivatives to zero, the second line gives $c_s = -\dfrac{\Omega_\mathrm{c}/2}{\delta + \mathrm{i}\gamma_s/2}\,c_e$. Inserting this into the first line:

$$0 = -(\Delta + \mathrm{i}\tfrac{\gamma}{2})c_e - \frac{\Omega_\mathrm{p}}{2} + \frac{\Omega_\mathrm{c}^2/4}{\delta + \mathrm{i}\gamma_s/2}\,c_e \quad \Rightarrow \quad \boxed{c_e = -\frac{\Omega_\mathrm{p}/2}{\Delta + \mathrm{i}\frac{\gamma}{2} - \dfrac{\Omega_\mathrm{c}^2/4}{\delta + \mathrm{i}\gamma_s/2}}}$$

Without control ($\Omega_\mathrm{c} = 0$) this is $c_e = -\frac{\Omega_\mathrm{p}/2}{\Delta + \mathrm{i}\gamma/2}$, the ordinary two-level response. The control field enters only through one extra term in the denominator. Everything in Part II follows from this one formula.

6 Electromagnetically induced transparency

How does c_e tell us what the gas does to the light? The atoms acquire an oscillating dipole, and many dipoles make a polarization that the light feels.

Susceptibility. The dipole of one atom (lab frame, $c_g = 1$) is $\langle\hat{d}\rangle = d(c_e\,\mathrm{e}^{-\mathrm{i}\omega_\mathrm{p}t} + \mathrm{c.c.})$. With $\mathcal{N}$ atoms per unit volume, the polarization is $P = \mathcal{N}\langle\hat{d}\rangle$. The susceptibility is defined by $P = \varepsilon_0\chi E$ for the parts oscillating as $\mathrm{e}^{-\mathrm{i}\omega_\mathrm{p}t}$, with $E = \frac{E_0}{2}(\mathrm{e}^{-\mathrm{i}\omega_\mathrm{p}t} + \mathrm{c.c.})$:

$$\mathcal{N}d\,c_e = \varepsilon_0\chi\,\frac{E_0}{2} \quad \Rightarrow \quad \chi = \frac{2\mathcal{N}d^2}{\varepsilon_0\hbar}\,\frac{c_e}{\Omega_\mathrm{p}} = -\frac{\mathcal{N}d^2}{\varepsilon_0\hbar}\,\frac{1}{\Delta + \mathrm{i}\frac{\gamma}{2} - \dfrac{\Omega_\mathrm{c}^2/4}{\delta + \mathrm{i}\gamma_s/2}},$$

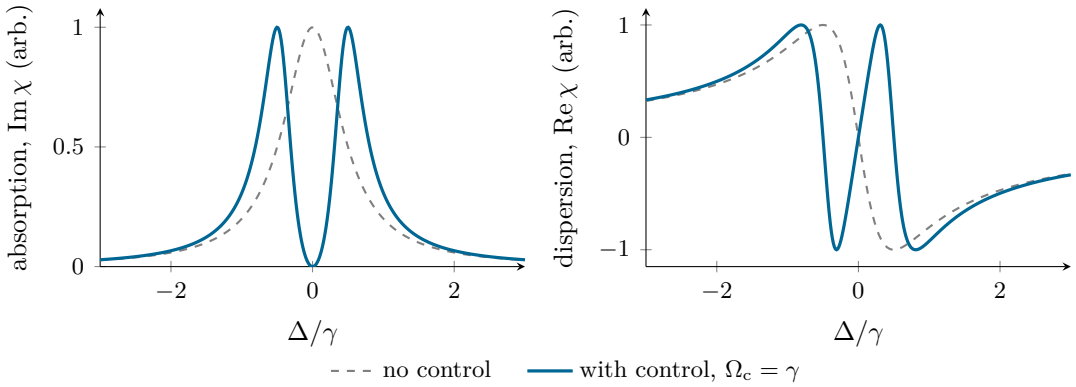
using $\Omega_\mathrm{p} = dE_0/\hbar$.

Propagation. The refractive index is $\sqrt{1+\chi} \approx 1 + \chi/2$, so $E \propto \mathrm{e}^{\mathrm{i}kz(1+\chi/2)}$ and the intensity falls as $\mathrm{e}^{-k\,\mathrm{Im}\chi\,z}$. So $\mathrm{Im}\,\chi$ is absorption and $\mathrm{Re}\,\chi$ gives the refractive index. Check with the two-level case: $\chi \propto -(\Delta - \mathrm{i}\frac{\gamma}{2})/(\Delta^2 + \frac{\gamma^2}{4})$, so $\mathrm{Im}\,\chi \propto \frac{\gamma/2}{\Delta^2+\gamma^2/4} > 0$, an absorption line of width γ.

Transparency. Now put the control on resonance with its transition, so that $\delta = \Delta$, and take the ideal case $\gamma_s = 0$:

$$\chi \propto -\frac{1}{\Delta + \mathrm{i}\frac{\gamma}{2} - \frac{\Omega_c^2}{4\Delta}}.$$

At $\Delta = 0$ the last term diverges, so $\chi = 0$: **no absorption and no dispersion at line centre.** For small Δ the last term dominates, $\chi \propto \frac{4\Delta}{\Omega_c^2}$, which is real. Near the centre the medium is transparent, and the refractive index rises steeply with Δ.



The absorption line gets a hole exactly at the centre, where the probe alone would be absorbed most strongly. This is **electromagnetically induced transparency** (EIT), first observed by Boller, Imamoğlu and Harris [2]. The physical reason is the dark state of Part I: with $\Omega_c \gg \Omega_p$, $|g\rangle$ is almost exactly $|D\rangle$, so the atom never goes up and no light is absorbed.

- With a strong control, $\Omega_c \gg \gamma$, the two absorption peaks sit at $\Delta = \pm\Omega_c/2$, where the real part of the denominator vanishes. This is the Autler–Townes splitting: the control dresses the excited state.
- With $\gamma_s \neq 0$ the transparency at the centre is no longer perfect. It stays good as long as $\Omega_c^2 \gg \gamma\gamma_s$ (see the exercises).

7 Slow light

A steep refractive index means slow light. A pulse envelope moves at the group velocity $v_g = \mathrm{d}\omega/\mathrm{d}k$. With $k = n(\omega)\,\omega/c$, differentiating gives $v_g = c/n_g$, with the group index $n_g = n + \omega\frac{\mathrm{d}n}{\mathrm{d}\omega}$.

Near the centre $\chi \approx \frac{\mathcal{N}d^2}{\varepsilon_0\hbar}\frac{4\Delta}{\Omega_c^2}$ and $n = 1 + \chi/2$; since $\frac{\mathrm{d}}{\mathrm{d}\omega} = \frac{\mathrm{d}}{\mathrm{d}\Delta}$ and the derivative term dominates,

$$n_g \approx \omega\frac{\mathrm{d}n}{\mathrm{d}\Delta} = \frac{\omega}{2}\cdot\frac{\mathcal{N}d^2}{\varepsilon_0\hbar}\cdot\frac{4}{\Omega_c^2}.$$

Express the density through the absorption coefficient *without* control at line centre, $\alpha_0 = k\,\mathrm{Im}\chi(0) = k\frac{\mathcal{N}d^2}{\varepsilon_0\hbar}\frac{2}{\gamma}$. With $\omega/k = c$:

Group velocity and delay

$$n_g \approx \frac{c\,\alpha_0\,\gamma}{\Omega_c^2} \quad\Rightarrow\quad v_g \approx \frac{\Omega_c^2}{\alpha_0\,\gamma}, \qquad \tau = \frac{L}{v_g} = \mathrm{OD}\cdot\frac{\gamma}{\Omega_c^2},$$

for a medium of length L and optical depth $\mathrm{OD} = \alpha_0 L$.

Numbers: rubidium, $\gamma/2\pi = 6\,\mathrm{MHz}$, $\Omega_c/2\pi = 3\,\mathrm{MHz}$, $\mathrm{OD} = 50$, $L = 1\,\mathrm{cm}$. Then $\tau \approx 5\ \mu\mathrm{s}$ and $v_g \approx 2\,\mathrm{km/s}$. Hau *et al.* slowed light to 17 m/s in an ultracold sodium gas [3].

Note the surprising sign: *weaker* control means *slower* light. If the control is switched off while the pulse is inside the medium, $v_g \to 0$ and the pulse stops. Its information is not lost: it

is stored in the coherence between $|g\rangle$ and $|s\rangle$ (the amplitude c_s), and switching the control back on releases the pulse [4, 5]. This is the basic idea of atomic quantum memories.

8 Far from resonance: Raman transitions

Last case: both lasers far from the excited state, $|\Delta| \gg \Omega_\mathrm{p}, \Omega_\mathrm{c}, \gamma$. The excited state is then hardly populated, yet the atom can move between $|g\rangle$ and $|s\rangle$. We eliminate $|e\rangle$ from the equations.

Adiabatic elimination. c_e responds fast (on a time $1/|\Delta|$) and follows the slowly varying c_g, c_s. So set $\dot{c}_e \approx 0$ in the rotating-frame equations:

$$0 = -\Delta\, c_e - \tfrac{\Omega_\mathrm{p}}{2} c_g - \tfrac{\Omega_\mathrm{c}}{2} c_s \quad \Rightarrow \quad c_e = -\frac{\Omega_\mathrm{p}\, c_g + \Omega_\mathrm{c}\, c_s}{2\Delta}.$$

Insert into the other two equations:

$$\mathrm{i}\dot{c}_g = \frac{\Omega_\mathrm{p}^2}{4\Delta} c_g + \frac{\Omega_\mathrm{p}\Omega_\mathrm{c}}{4\Delta} c_s, \qquad \mathrm{i}\dot{c}_s = \Big(\frac{\Omega_\mathrm{c}^2}{4\Delta} - \delta\Big) c_s + \frac{\Omega_\mathrm{p}\Omega_\mathrm{c}}{4\Delta} c_g.$$

These are the equations of a two-level system made of the two ground states.

Effective two-level atom

$$\hat{H}_\mathrm{eff} = \begin{pmatrix} \frac{\Omega_\mathrm{p}^2}{4\Delta} & \frac{\Omega_\mathrm{R}}{2} \\ \frac{\Omega_\mathrm{R}}{2} & \frac{\Omega_\mathrm{c}^2}{4\Delta} - \delta \end{pmatrix}, \qquad \Omega_\mathrm{R} = \frac{\Omega_\mathrm{p}\Omega_\mathrm{c}}{2\Delta}.$$

- Off-diagonal: the two-photon **Raman Rabi frequency** Ω_R.
- Diagonal: **light shifts** $\frac{\Omega_\mathrm{p}^2}{4\Delta}$ and $\frac{\Omega_\mathrm{c}^2}{4\Delta}$; each laser shifts the level it couples to.
- Resonance when the diagonal elements are equal: $\delta = \frac{\Omega_\mathrm{c}^2 - \Omega_\mathrm{p}^2}{4\Delta}$.

Why go far from resonance? The atom spends only a fraction $|c_e|^2 \sim \Omega^2/4\Delta^2$ of its time in $|e\rangle$, so it scatters photons at a rate $\sim \gamma\Omega^2/4\Delta^2$, while the coherent Raman rate is $\sim \Omega^2/\Delta$. Their ratio is $\sim \gamma/\Delta$: far detuning makes the transfer almost perfectly coherent. Two optical lasers then drive a GHz transition between long-lived states. This is how ground-state qubits are manipulated in many atomic and ionic systems, and spontaneous Raman scattering is how atomic memories create pairs of a photon and a stored excitation [6].

Summary

1. RWA: drop terms oscillating at the sum of laser and atomic frequencies (and the "wrong" laser on each transition). Valid when $\Omega, \Delta \ll$ those frequencies. The rotating frame that follows is exact.
2. Λ system in the rotating frame: a 3×3 matrix with one-photon detuning Δ and two-photon detuning δ.
3. At $\delta = 0$ the state $|D\rangle \propto \Omega_\mathrm{c}|g\rangle - \Omega_\mathrm{p}|s\rangle$ is dark: the two paths to $|e\rangle$ cancel.
4. Spontaneous emission pumps atoms into $|D\rangle$: coherent population trapping.
5. A weak probe sees $\chi \propto -1/\big[\Delta + \mathrm{i}\frac{\gamma}{2} - \frac{\Omega_\mathrm{c}^2/4}{\delta + \mathrm{i}\gamma_s/2}\big]$: a transparency window at $\delta = 0$.
6. The steep dispersion gives $v_g \approx \Omega_\mathrm{c}^2/(\alpha_0\gamma)$: slow, and even stopped, light.
7. Far from resonance, $|e\rangle$ drops out: Rabi frequency $\Omega_\mathrm{p}\Omega_\mathrm{c}/2\Delta$ between the ground states, plus light shifts.

References

[1] G. Alzetta, A. Gozzini, L. Moi, G. Orriols, Nuovo Cimento B **36**, 5 (1976).

[2] K.-J. Boller, A. Imamoğlu, S. E. Harris, Phys. Rev. Lett. **66**, 2593 (1991).

[3] L. V. Hau, S. E. Harris, Z. Dutton, C. H. Behroozi, Nature **397**, 594 (1999).

[4] M. Fleischhauer and M. D. Lukin, Phys. Rev. Lett. **84**, 5094 (2000).

[5] C. Liu, Z. Dutton, C. H. Behroozi, L. V. Hau, Nature **409**, 490 (2001).

[6] M. Parniak, *Multimode Quantum Optics with Spin Waves and Photons*, PhD thesis, University of Warsaw (2019), Sec. 2.3.

[7] Further reading: M. Fleischhauer, A. Imamoğlu, J. P. Marangos, Rev. Mod. Phys. **77**, 633 (2005).