Generating functions

Photon statistics, losses and the beam splitter

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Conventions. $\hat{n} = \hat{a}^\dagger\hat{a}$; p_n is the probability to count n photons; $g^{(2)} = \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle/\langle\hat{a}^\dagger\hat{a}\rangle^2$. Background: lecture notes "Nonclassical light I: photon counting".

Exercise 1: The generating function: what it is and what it gives

Useful to remember

- p_n is the probability to count n photons; $\sum_n p_n = 1$, $\langle n\rangle = \sum_n n\,p_n$.
- From the lectures: $g^{(2)} = \dfrac{\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2}$, and since $\hat{a}^{\dagger 2}\hat{a}^2 = \hat{n}(\hat{n}-1)$, $g^{(2)} = \dfrac{\langle n(n-1)\rangle}{\langle n\rangle^2}$. Coherent light $g^{(2)} = 1$, thermal $g^{(2)} = 2$, a single photon $g^{(2)} = 0$.
- Coherent state: $p_n = \mathrm{e}^{-N}N^n/n!$ (Poisson). Thermal state: $p_n = (1-\lambda)\lambda^n$ (geometric), $0 < \lambda < 1$.
- Two series: $\sum_n \frac{z^n}{n!} = \mathrm{e}^z$, $\sum_n z^n = \frac{1}{1-z}$ for $|z| < 1$.

Define the **generating function** of the photon-number distribution

$$G(x) \equiv \sum_{n=0}^{\infty} p_n\, x^n, \qquad -1 \le x \le 1.$$

(a) Show that $G(1) = 1$, $G'(1) = \langle n\rangle$, $G''(1) = \langle n(n-1)\rangle$, and $p_n = \frac{1}{n!}\,G^{(n)}(0)$.

(b) Express $\mathrm{Var}(n)$ and $g^{(2)}$ through $G'(1)$ and $G''(1)$.

(c) Compute $G(x)$ for a coherent state. Find $\langle n\rangle$, $\mathrm{Var}(n)$ and $g^{(2)}$.

(d) Thermal state. First show that $\langle n\rangle = \lambda/(1-\lambda)$, so that $\lambda = N/(1+N)$ with $N \equiv \langle n\rangle$. Then show

$$G(x) = \frac{1}{1+N(1-x)},$$

and find $\mathrm{Var}(n)$ and $g^{(2)}$.

(e) Fock state $|m\rangle$: find $G(x)$ and $g^{(2)}$.

(f) A state is a mixture: with probability q it is $\hat{\rho}_1$, otherwise $\hat{\rho}_2$. How is its G related to G_1 and G_2?

Solution

(a) At $x = 1$ all powers equal 1: $G(1) = \sum_n p_n = 1$. Differentiate term by term:

$$G'(x) = \sum_n n\,p_n\,x^{n-1}, \qquad G''(x) = \sum_n n(n-1)\,p_n\,x^{n-2}.$$

At $x = 1$: $G'(1) = \sum_n n\,p_n = \langle n\rangle$ and $G''(1) = \sum_n n(n-1)p_n = \langle n(n-1)\rangle$.
Differentiating k times kills all terms with $n < k$; at $x = 0$ all terms with $n > k$ vanish too. The single surviving term is $k!\,p_k$, so $p_k = G^{(k)}(0)/k!$. In words: G is the Taylor series whose coefficients are the probabilities.

(b) $\langle n^2\rangle = \langle n(n-1)\rangle + \langle n\rangle = G''(1) + G'(1)$, so

$$\mathrm{Var}(n) = \langle n^2\rangle - \langle n\rangle^2 = G''(1) + G'(1) - G'(1)^2, \qquad g^{(2)} = \frac{\langle n(n-1)\rangle}{\langle n\rangle^2} = \frac{G''(1)}{G'(1)^2}.$$

(c) Coherent state:

$$G(x) = \sum_n \mathrm{e}^{-N}\frac{N^n}{n!}x^n = \mathrm{e}^{-N}\sum_n \frac{(Nx)^n}{n!} = \mathrm{e}^{-N}\mathrm{e}^{Nx} = \mathrm{e}^{-N(1-x)}.$$

$G'(x) = N\,\mathrm{e}^{-N(1-x)}$ and $G''(x) = N^2\mathrm{e}^{-N(1-x)}$, so $G'(1) = N$, $G''(1) = N^2$. Hence

$$\langle n\rangle = N, \qquad \mathrm{Var}(n) = N^2 + N - N^2 = N, \qquad g^{(2)} = \frac{N^2}{N^2} = 1.$$

(d) Thermal state. First the generating function in terms of λ, using the geometric series with $z = \lambda x$:

$$G(x) = (1-\lambda)\sum_n (\lambda x)^n = \frac{1-\lambda}{1-\lambda x}.$$

Mean: $G'(x) = \frac{(1-\lambda)\lambda}{(1-\lambda x)^2}$, so $\langle n\rangle = G'(1) = \frac{\lambda}{1-\lambda} \equiv N$. Solving, $\lambda = \frac{N}{1+N}$ and $1-\lambda = \frac{1}{1+N}$. Insert:

$$G(x) = \frac{\frac{1}{1+N}}{1-\frac{Nx}{1+N}} = \frac{1}{1+N-Nx} = \frac{1}{1+N(1-x)}.$$

Derivatives, writing $D = 1 + N(1-x)$ (so $D' = -N$):

$$G'(x) = \frac{N}{D^2}, \qquad G''(x) = \frac{2N^2}{D^3}; \qquad \text{at } x = 1,\ D = 1: \quad G'(1) = N, \quad G''(1) = 2N^2.$$

$$\mathrm{Var}(n) = 2N^2 + N - N^2 = N + N^2, \qquad g^{(2)} = \frac{2N^2}{N^2} = 2.$$

Thermal light is noisier than Poissonian: the extra N^2 is the bunching.

(e) $p_n = \delta_{nm}$, so $G(x) = x^m$. Then $G'(1) = m$, $G''(1) = m(m-1)$, and

$$g^{(2)} = \frac{m(m-1)}{m^2} = 1 - \frac{1}{m}, \qquad \mathrm{Var}(n) = m(m-1) + m - m^2 = 0.$$

For $m = 1$: $g^{(2)} = 0$, the single photon.

(f) The probabilities mix linearly, $p_n = q\,p_n^{(1)} + (1-q)\,p_n^{(2)}$, and so does G:

$$G(x) = q\,G_1(x) + (1-q)\,G_2(x).$$

(Note: $g^{(2)}$ does *not* mix linearly, because it is a ratio.)

Exercise 2: Losses and the beam splitter

Useful to remember

- A loss of transmission η (absorber, imperfect detector) is a beam splitter whose second output is thrown away. For photon counting each photon behaves independently: it is transmitted with probability η and lost with probability $1-\eta$. (This follows from $\hat{a} \to \sqrt{\eta}\,\hat{a} + \sqrt{1-\eta}\,\hat{v}$ with vacuum in $\hat{v}$; we take it as given.)
- Binomial theorem: $\sum_{m=0}^{n}\binom{n}{m}a^m b^{n-m} = (a+b)^n$.

- Hanbury Brown–Twiss (HBT): split the light on a beam splitter and count coincidences between the two outputs.

(a) n photons fall on a loss η. What is the probability that m of them get through? Show that the generating function after the loss is

$$G_{\text{out}}(x) = G(1 - \eta + \eta x).$$

(b) What happens to a coherent state? To a thermal state? To a single photon?

(c) Using the chain rule, show that $g^{(2)}$ does not change under loss. Show that the Fano factor $F \equiv \text{Var}(n)/\langle n\rangle$ becomes

$$F_{\text{out}} = 1 + \eta\,(F - 1).$$

What does loss do to sub-Poissonian ($F < 1$) light?

(d) A beam splitter with transmission T and reflection $R = 1 - T$ (vacuum in the second input) sends each photon to output c or d. Define $G_{cd}(x, y) \equiv \sum_{k,l} p_{kl}\, x^k y^l$ for the joint distribution of the two outputs. Show that $G_{cd}(x, y) = G(Tx + Ry)$.

(e) Show that $\langle n_c n_d\rangle/(\langle n_c\rangle\langle n_d\rangle) = g^{(2)}$ of the input light. Why is this a good way of measuring $g^{(2)}$?

Solution

(a) Each of the n photons is transmitted independently with probability η, so the number transmitted is binomial:

$$P(m|n) = \binom{n}{m}\eta^m(1-\eta)^{n-m}.$$

The output distribution is $p'_m = \sum_n p_n\, P(m|n)$. Now compute its generating function; the binomial theorem does the sum over m:

$$G_{\text{out}}(x) = \sum_m p'_m\, x^m = \sum_n p_n \sum_{m=0}^{n} \binom{n}{m}(\eta x)^m(1-\eta)^{n-m}$$

$$= \sum_n p_n\,(1 - \eta + \eta x)^n = G(1 - \eta + \eta x).$$

The whole effect of a loss is a substitution of the argument.
(b) Coherent: $1 - (1 - \eta + \eta x) = \eta(1 - x)$, so $G_{\text{out}} = \mathrm{e}^{-\eta N(1-x)}$: still coherent, with $N \to \eta N$. Thermal: $G_{\text{out}} = \frac{1}{1+\eta N(1-x)}$: still thermal, with $N \to \eta N$. Single photon: $G_{\text{out}} = 1 - \eta + \eta x$, i.e. $p_0 = 1 - \eta$, $p_1 = \eta$: the photon is either there or lost.
(c) With $u = 1 - \eta + \eta x$, $\frac{\mathrm{d}u}{\mathrm{d}x} = \eta$, and $u = 1$ at $x = 1$:

$$G'_{\text{out}}(1) = \eta\, G'(1), \qquad G''_{\text{out}}(1) = \eta^2\, G''(1) \quad \Rightarrow \quad g^{(2)}_{\text{out}} = \frac{\eta^2 G''(1)}{\eta^2 G'(1)^2} = g^{(2)}.$$

For the variance, use 1(b) with $\langle n\rangle = G'(1)$ and $G''(1) = \text{Var}(n) - \langle n\rangle + \langle n\rangle^2$:

$$\text{Var}_{\text{out}} = \eta^2 G''(1) + \eta G'(1) - \eta^2 G'(1)^2 = \eta^2\big[\,\text{Var}(n) - \langle n\rangle\big] + \eta\langle n\rangle.$$

Dividing by $\langle n\rangle_{\text{out}} = \eta\langle n\rangle$:

$$F_{\text{out}} = \eta\,(F - 1) + 1.$$

Loss moves F towards 1 (Poisson) linearly in η. Sub-Poissonian light stays sub-Poissonian, but the effect shrinks; the single photon ($F = 0$) becomes $F_{\text{out}} = 1 - \eta$.

(d) Each photon goes to c with probability T and to d with probability R. For n photons the probability of the split (k, l) with $k + l = n$ is $\frac{n!}{k!\,l!} T^k R^l$. Therefore

$$G_{cd}(x, y) = \sum_n p_n \sum_{k+l=n} \frac{n!}{k!\,l!} (Tx)^k (Ry)^l = \sum_n p_n \,(Tx + Ry)^n = G(Tx + Ry).$$

Check: setting $y = 1$ means we ignore output d. Then $G_{cd}(x, 1) = G(Tx + 1 - T)$, which is a loss with $\eta = T$, as it must be.

(e) Joint moments come from mixed derivatives: $\langle n_c n_d \rangle = \partial_x \partial_y G_{cd}\big|_{x=y=1}$. With $u = Tx + Ry$:

$$\partial_y G_{cd} = R\,G'(u), \qquad \partial_x \partial_y G_{cd} = TR\,G''(u) \;\Rightarrow\; \langle n_c n_d \rangle = TR\,G''(1).$$

Also $\langle n_c \rangle = T\,G'(1)$ and $\langle n_d \rangle = R\,G'(1)$. So

$$\frac{\langle n_c n_d \rangle}{\langle n_c \rangle \langle n_d \rangle} = \frac{TR\,G''(1)}{TR\,G'(1)^2} = g^{(2)}.$$

Coincidences between two detectors measure $g^{(2)}$. Each detector only needs to see one photon at a time, and by (c) losses in front of the detectors do not change the result. For a single photon $G'' = 0$: there are never coincidences, the photon is not split.