Nonclassical light I: photon counting

Coherent states, photodetection, the P function, generating functions and $g^{(2)}$

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About these notes

These notes follow the blackboard lecture. The guiding question is: *what makes a state of light nonclassical?* Here we answer it with photon counting. Every result is derived; the exercises (sheet "Generating functions") redo the most important calculations step by step.

Conventions. $\hbar = 1$. One mode of the field with annihilation operator $\hat{a}$, $[\hat{a}, \hat{a}^\dagger] = 1$, photon number $\hat{n} = \hat{a}^\dagger\hat{a}$, Fock states $|n\rangle$. Quadratures $\hat{x} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$, $\hat{p} = (\hat{a} - \hat{a}^\dagger)/(\mathrm{i}\sqrt{2})$; in the vacuum $\operatorname{Var}\hat{x} = \operatorname{Var}\hat{p} = \frac{1}{2}$. For a complex number α we write $\mathrm{d}^2\alpha = \mathrm{d}(\operatorname{Re}\alpha)\,\mathrm{d}(\operatorname{Im}\alpha)$.

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1 Coherent states

Coherent states describe an ideal laser, and everything in these notes is built on them. We collect their properties, each with a short proof.

Definition. For a complex number α,

$$|\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle, \qquad \hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right).$$

$\hat{D}$ is unitary (the exponent is anti-Hermitian) and is called the *displacement operator*.

Displacement. Let $X = \alpha\hat{a}^\dagger - \alpha^*\hat{a}$. Then $[X, \hat{a}] = -\alpha$ is a number, so the series $\mathrm{e}^{-X}\hat{a}\,\mathrm{e}^{X} = \hat{a} - [X, \hat{a}] + \frac{1}{2}[X, [X, \hat{a}]] - \dots$ stops after two terms:

$$\hat{D}^\dagger\hat{a}\,\hat{D} = \hat{a} + \alpha.$$

Eigenstate of $\hat{a}$. Using $\hat{D}\hat{D}^\dagger = \mathbb{1}$ and $\hat{a}|0\rangle = 0$:

$$\hat{a}|\alpha\rangle = \hat{D}\,(\hat{D}^\dagger\hat{a}\hat{D})|0\rangle = \hat{D}(\hat{a} + \alpha)|0\rangle = \alpha|\alpha\rangle.$$

Fock expansion. Write $|\alpha\rangle = \sum_n c_n|n\rangle$. Since $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, the eigenvalue equation gives $\sqrt{n+1}\,c_{n+1} = \alpha\, c_n$, hence $c_n = c_0\,\alpha^n/\sqrt{n!}$. Normalization fixes c_0:

$$\boxed{|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle}, \qquad \langle\alpha|\beta\rangle = \mathrm{e}^{-\frac{|\alpha|^2+|\beta|^2}{2}+\alpha^*\beta}, \qquad |\langle\alpha|\beta\rangle|^2 = \mathrm{e}^{-|\alpha-\beta|^2}.$$

So the photon number in $|\alpha\rangle$ is Poisson distributed with mean $|\alpha|^2$. Coherent states are not orthogonal.

Completeness. Write $\alpha = r\mathrm{e}^{\mathrm{i}\phi}$, $\mathrm{d}^2\alpha = r\,\mathrm{d}r\,\mathrm{d}\phi$. The ϕ-integral of $\alpha^n\alpha^{*m}$ vanishes unless $n = m$, and $\int_0^\infty 2r\,\mathrm{d}r\,r^{2n}\mathrm{e}^{-r^2} = n!$. Therefore

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha| = \sum_{n,m}\frac{|n\rangle\langle m|}{\pi\sqrt{n!\,m!}}\int \mathrm{d}^2\alpha\,\mathrm{e}^{-|\alpha|^2}\alpha^n\alpha^{*m} = \sum_n |n\rangle\langle n| = \mathbb{1}.$$

Even though they are not orthogonal, coherent states can be used to expand any state or operator.

Physical picture. $\langle\hat{x}\rangle = \sqrt{2}\,\mathrm{Re}\,\alpha$ and $\langle\hat{p}\rangle = \sqrt{2}\,\mathrm{Im}\,\alpha$, while the variances are those of the vacuum ($\hat{D}$ only shifts $\hat{a}$ by a number). A coherent state is a classical wave of complex amplitude α plus vacuum noise.

Beam splitter. A beam splitter with transmission T and reflection $R = 1 - T$ maps $|\alpha\rangle|0\rangle \to |\sqrt{T}\alpha\rangle|\sqrt{R}\alpha\rangle$ (up to phases of the amplitudes). The output is a *product* state: a coherent state splits like a classical wave. Discarding one output port gives the model of a loss: $|\alpha\rangle \to |\sqrt{\eta}\,\alpha\rangle$ with transmission η.

2 Semiclassical photodetection

We first count photons *without photons*: a classical field falls on a quantum detector. This tells us which observations a classical field can explain.

Model. A classical field of intensity I falls on a detector for a time Δt. The atoms of the detector are ionized independently; by Fermi's golden rule the probability of one event in a short interval δt is $CI\,\delta t$, with C a constant.

Constant intensity: Poisson. Cut Δt into N short slices. Each slice gives a count with probability W/N, where $W \equiv CI\,\Delta t$. The number of counts is binomial, and for $N \to \infty$ it becomes Poissonian:

$$p_n = \binom{N}{n}\Big(\frac{W}{N}\Big)^n\Big(1-\frac{W}{N}\Big)^{N-n} \xrightarrow{N\to\infty} \frac{W^n}{n!}\,\mathrm{e}^{-W}, \qquad \langle n\rangle = W, \quad \mathrm{Var}(n) = W.$$

Even a perfectly stable wave gives fluctuating counts. This is *shot noise*.

Fluctuating intensity: the Mandel formula. If W changes from shot to shot with probability density $\mathcal{P}(W)$, we average the Poisson law:

$$\boxed{p_n = \int_0^\infty \mathrm{d}W\;\mathcal{P}(W)\,\frac{W^n}{n!}\,\mathrm{e}^{-W}}$$

For a Poisson law $\langle n\rangle = W$ and $\langle n^2\rangle = W + W^2$. Averaging over W (bar):

$$\mathrm{Var}(n) = \overline{W} + \overline{W^2} - \overline{W}^{\,2} = \langle n\rangle + \mathrm{Var}(W) \;\geq\; \langle n\rangle.$$

A fluctuating classical wave can only *add* noise to the shot noise. Counts with $\mathrm{Var}(n) < \langle n\rangle$, called *sub-Poissonian*, cannot be produced by any classical field.

Thermal light

Why the intensity is exponentially distributed. The field in one mode of thermal light is a sum of contributions of many independent emitters with random phases,

$$E = \sum_{k=1}^{K} a_k \, \mathrm{e}^{\mathrm{i}\varphi_k}, \qquad \varphi_k \text{ independent and uniform on } [0, 2\pi).$$

Uniform phases make $\operatorname{Re} E$ and $\operatorname{Im} E$ zero-mean, of equal variance, and uncorrelated (because $\overline{\cos\varphi \sin\varphi} = 0$ and $\overline{\cos^2\varphi} = \overline{\sin^2\varphi} = \frac{1}{2}$). For $K \gg 1$ the central limit theorem makes E a circular complex Gaussian variable; with $\sigma^2 \equiv \overline{|E|^2}$,

$$p(E)\,\mathrm{d}^2E = \frac{1}{\pi\sigma^2}\,\mathrm{e}^{-|E|^2/\sigma^2}\,\mathrm{d}^2E.$$

In polar coordinates $E = r\mathrm{e}^{\mathrm{i}\phi}$ we have $\mathrm{d}^2E = r\,\mathrm{d}r\,\mathrm{d}\phi = \frac{1}{2}\,\mathrm{d}(r^2)\,\mathrm{d}\phi$. Integrating out ϕ and using $W \propto r^2$:

$$\mathcal{P}(W) = \frac{1}{\bar{W}}\,\mathrm{e}^{-W/\bar{W}}.$$

The squared length of a two-dimensional Gaussian vector is exponentially distributed. This is the same physics as laser speckle.

Counts. With $\int_0^\infty \mathrm{d}W\, W^n \mathrm{e}^{-\lambda W} = n!/\lambda^{n+1}$ the Mandel formula gives

$$p_n = \frac{1}{\bar{W}\,n!}\int_0^\infty \mathrm{d}W\, W^n \mathrm{e}^{-W(1+1/\bar{W})} = \frac{\bar{W}^n}{(1+\bar{W})^{n+1}}, \qquad \mathrm{Var}(n) = \bar{W} + \bar{W}^2,$$

where we used $\overline{W^2} = 2\bar{W}^2$ for the exponential distribution. This is the Bose–Einstein distribution, obtained without photons. The extra noise $\bar{W}^2$ comes from the chaotic intensity fluctuations.

Two remarks

- **Cross-check.** For a single mode in thermal equilibrium the energy is proportional to $|E|^2$, so the Boltzmann factor gives $\mathrm{e}^{-W/\bar{W}}$ directly. The exponential is also the maximum-entropy distribution at a fixed mean intensity.
- **Single mode only.** This requires a detection time shorter than the coherence time and a detector smaller than the coherence area. A detector that integrates over M independent modes sees a sum of M exponentially distributed intensities, which is less noisy.

3 The P function and the definition of classical light

Quantum mechanically, the detector measures $\hat{n}$ and $p_n = \langle n|\hat{\rho}|n\rangle$. To compare with the semi-classical theory, we write the state as a mixture of classical waves, that is, of coherent states.

The P representation

Glauber and Sudarshan [1, 2]: every state can be written as

$$\hat{\rho} = \int \mathrm{d}^2\alpha\; P(\alpha)\,|\alpha\rangle\langle\alpha|, \qquad \int \mathrm{d}^2\alpha\, P(\alpha) = 1.$$

P is real, but in general it is a *generalized function*: it can be negative, and more singular than a δ-function.

Definition. $\hat{\rho}$ is *classical* if $P \geq 0$ (a probability density; δ-functions allowed). Otherwise it is *nonclassical*.

Normally ordered moments. An operator product is *normally ordered* if all $\hat{a}^\dagger$ stand to the left of all $\hat{a}$. Using $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$:

$$\langle \hat{a}^{\dagger k}\hat{a}^l\rangle = \int \mathrm{d}^2\alpha\, P(\alpha)\,\langle\alpha|\hat{a}^{\dagger k}\hat{a}^l|\alpha\rangle = \int \mathrm{d}^2\alpha\, P(\alpha)\,\alpha^{*k}\alpha^l.$$

If $P \geq 0$, the state is a statistical mixture of classical waves with random amplitude α, and every normally ordered moment is a classical average. The vacuum and coherent states count as classical in this optical sense.

A *nonclassicality witness* is a measurable quantity whose value is impossible if $P \geq 0$. Three examples follow.

Coherent state $|\alpha_0\rangle$: $P = \delta^2(\alpha - \alpha_0)$. Classical.

Thermal state $\hat{\rho}_{\mathrm{th}} = \sum_n \frac{N^n}{(1+N)^{n+1}}|n\rangle\langle n|$, $P = \frac{1}{\pi N}\mathrm{e}^{-|\alpha|^2/N}$. To check this, use the Fock expansion of $|\alpha\rangle$. Off-diagonal elements $\langle n|\hat{\rho}|m\rangle \propto \int \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}(n-m)\phi}$ vanish for $n \neq m$. For $n = m$, with $W = |\alpha|^2$ and $\mathrm{d}^2\alpha = \pi\,\mathrm{d}W$ after the angular integral,

$$\langle n|\hat{\rho}|n\rangle = \int \mathrm{d}^2\alpha\, P(\alpha)\,\frac{|\alpha|^{2n}}{n!}\mathrm{e}^{-|\alpha|^2} = \int_0^\infty \mathrm{d}W\,\frac{\mathrm{e}^{-W/N}}{N}\,\frac{W^n}{n!}\mathrm{e}^{-W} = \frac{N^n}{(1+N)^{n+1}}.$$

This is exactly the chaotic wave of Section 2. Classical.

Fock state $|m\rangle$, $m \geq 1$. Since $\hat{n}^2 = \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} = \hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)\hat{a} = \hat{a}^{\dagger 2}\hat{a}^2 + \hat{n}$, if $P \geq 0$ we could write

$$\mathrm{Var}(n) = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle + \langle \hat{a}^\dagger\hat{a}\rangle - \langle \hat{a}^\dagger\hat{a}\rangle^2 = \langle n\rangle + \Big[\int P\,|\alpha|^4 - \Big(\int P\,|\alpha|^2\Big)^2\Big] \;\geq\; \langle n\rangle = m,$$

because the bracket is the variance of $|\alpha|^2$ over a probability distribution. But in $|m\rangle$ the variance is 0. So $P \geq 0$ is impossible: **Fock states are nonclassical.**

This is the general strategy in miniature: derive an inequality valid for every $P \geq 0$, then find states that violate it.

4 The generating function

Definition and basic properties

$$G(x) \equiv \sum_{n=0}^{\infty} p_n\, x^n = \mathrm{Tr}\left[\hat{\rho}\, x^{\hat{n}}\right], \qquad -1 \leq x \leq 1.$$

(i) $G(1) = 1$, and $p_n = \frac{1}{n!}G^{(n)}(0)$.
(ii) $G'(1) = \langle n\rangle$, $G''(1) = \langle n(n-1)\rangle = \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle$ (from $\frac{\mathrm{d}^2}{\mathrm{d}x^2}x^n = n(n-1)x^{n-2}$ and $\hat{n}^2 - \hat{n} = \hat{a}^{\dagger 2}\hat{a}^2$).
(iii) $G(-1) = \sum_n (-1)^n p_n = \langle(-1)^{\hat{n}}\rangle$, the photon-number parity.
(iv) An on/off detector of efficiency η misses each photon with probability $1-\eta$, so $p(\text{no click}) = G(1-\eta)$.

G from P. In a coherent state $\langle\alpha|x^{\hat{n}}|\alpha\rangle = \sum_n x^n\,\mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = \mathrm{e}^{-(1-x)|\alpha|^2}$. Therefore

$$\boxed{G(x) = \int \mathrm{d}^2\alpha\; P(\alpha)\,\mathrm{e}^{-(1-x)|\alpha|^2}}$$

Expanding in powers of x gives back the Mandel formula with $W = |\alpha|^2$. So the semiclassical theory of Section 2 is *exact* for classical light, and for nonclassical light it still holds formally, with a weight P that is not positive.

Loss. A loss of transmission η maps each $|\alpha\rangle$ in the P representation to $|\sqrt{\eta}\,\alpha\rangle$, so

$$G_{\text{out}}(x) = \int \mathrm{d}^2\alpha\, P(\alpha)\, \mathrm{e}^{-(1-x)\eta|\alpha|^2} = G(1-\eta+\eta x).$$

Beam splitter. Let both outputs c, d of a T:R beam splitter be counted, and define $G_{cd}(x,y) \equiv \mathrm{Tr}[\hat{\rho}\, x^{\hat{n}_c} y^{\hat{n}_d}]$. Each term becomes $|\sqrt{T}\alpha\rangle|\sqrt{R}\alpha\rangle$, a product, so

$$G_{cd}(x,y) = \int \mathrm{d}^2\alpha\, P(\alpha)\, \mathrm{e}^{-(1-x)T|\alpha|^2} \mathrm{e}^{-(1-y)R|\alpha|^2} = G_a(Tx+Ry).$$

Expanding $(Tx+Ry)^n$ with the binomial theorem shows the meaning: every photon independently goes to c with probability T and to d with probability R. This holds for any input state.

Catalogue

coherent ($P = \delta^2(\alpha-\alpha_0)$, $N = \lvert\alpha_0\rvert^2$):	$G = \mathrm{e}^{-N(1-x)}$
thermal ($P = \frac{1}{\pi N}\mathrm{e}^{-\lvert\alpha\rvert^2/N}$):	$G = \frac{1}{\pi N}\int \mathrm{d}^2\alpha\, \mathrm{e}^{-\lvert\alpha\rvert^2[\frac{1}{N}+1-x]} = \dfrac{1}{1+N(1-x)}$
Fock $\lvert m\rangle$:	$G = x^m$

(The thermal integral uses $\int \mathrm{d}^2\alpha\, \mathrm{e}^{-c|\alpha|^2} = \pi/c$.) Under a loss, coherent and thermal G keep their form with $N \to \eta N$. For light made of independent modes, the generating functions multiply.

5 Second-order coherence and the Hanbury Brown–Twiss experiment

$g^{(2)}$

$$g^{(2)} \equiv \frac{\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle}{\langle \hat{a}^\dagger\hat{a}\rangle^2} = \frac{G''(1)}{G'(1)^2}$$

coherent: 1, thermal: $\dfrac{2N^2}{N^2} = 2$, Fock $|m\rangle$: $\dfrac{m(m-1)}{m^2} = 1 - \dfrac{1}{m}$.

Classical bound. If $P \geq 0$,

$$\langle \hat{a}^{\dagger 2}\hat{a}^2\rangle - \langle \hat{a}^\dagger\hat{a}\rangle^2 = \int \mathrm{d}^2\alpha\, P(\alpha)\big(|\alpha|^2 - \langle n\rangle\big)^2 \geq 0 \quad \Rightarrow \quad \boxed{g^{(2)} \geq 1 \text{ for classical light.}}$$

$g^{(2)} < 1$ is called *antibunching*; it is a nonclassicality witness. It is equivalent to sub-Poissonian counts, since $\mathrm{Var}(n) - \langle n\rangle = \langle n\rangle^2(g^{(2)}-1)$.

Loss. The derivatives of $G(1-\eta+\eta x)$ at $x=1$ are $\eta G'(1)$ and $\eta^2 G''(1)$, so $g^{(2)}$ does not change under loss.

Hanbury Brown–Twiss (HBT). Split the light on a beam splitter and correlate the counts of the two outputs. From $G_{cd} = G_a(Tx+Ry)$:

$$\langle n_c n_d\rangle = \partial_x\partial_y G_{cd}\big|_{1,1} = TR\,G_a''(1), \qquad \langle n_c\rangle\langle n_d\rangle = TR\,G_a'(1)^2 \quad \Rightarrow \quad \frac{\langle n_c n_d\rangle}{\langle n_c\rangle\langle n_d\rangle} = g_a^{(2)}.$$

So $g^{(2)}$ is measured by coincidences after a beam splitter, even with detectors that cannot count photons, and it does not depend on losses. That is why it is the standard figure of merit for single-photon sources. A single photon gives no coincidences at all: it is not split. Antibunching was first observed by Kimble, Dagenais and Mandel [3].

Summary

state	p_n	$G(x)$	$\mathrm{Var}(n)$	$g^{(2)}$	classical?
coherent	Poisson	$\mathrm{e}^{-N(1-x)}$	N	1	yes
thermal	Bose–Einstein	$\frac{1}{1+N(1-x)}$	$N+N^2$	2	yes
Fock $\lvert m\rangle$	δ_{nm}	x^m	0	$1-\frac{1}{m}$	no

- Classical light is a mixture of coherent states with a positive weight P. For it, counting statistics are those of a fluctuating classical wave, and $\mathrm{Var}(n) \geq \langle n\rangle$, $g^{(2)} \geq 1$.
- The generating function $G(x)$ packs the whole photon-number distribution into one function. Moments are derivatives at $x=1$, a loss is the substitution $x \to 1-\eta+\eta x$, and a beam splitter is $G(Tx+Ry)$.

References

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[3] H. J. Kimble, M. Dagenais, L. Mandel, Phys. Rev. Lett. **39**, 691 (1977).

[4] Further reading: J. W. Goodman, *Statistical Optics* (Wiley 1985), Ch. 2 and 6; C. Gerry, P. Knight, *Introductory Quantum Optics* (CUP 2005), Ch. 3 and 7; W. Wasilewski, PhD thesis, UMK Toruń (2007), Ch. 1 and 4; M. Parniak, *Multimode Quantum Optics with Spin Waves and Photons*, PhD thesis, University of Warsaw (2019), Sec. 2.1.